Math 210B Lecture 13 Notes

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1 Hom-$\otimes$ Adjunction, Tensor Powers, and Graded Algebras

1.1 Adjunction of Hom and $\otimes$

Theorem 1.1. *Let A, B, C be R-algebras, and let M, N, L be R-balanced A-B, B-C, and A-C bimodules, respectively. Then $\operatorname{Hom}_A(M \otimes_B N, L) \cong \operatorname{Hom}_B(N, \operatorname{Hom}_A(M, L))$ as right C-modules. Moreover, these are natural in M, N, L. In fact, we have $t_M : B \otimes_R C^{\mathrm{op}}\text{-mod} \to A \otimes_R C^{\mathrm{op}}\text{-mod}$*

$$\begin{array}{ccc} N & \longrightarrow & M \otimes_R N \\ \big\downarrow{\scriptstyle \lambda} & & \big\downarrow{\scriptstyle \mathrm{id}_M \otimes_R \lambda} \\ N' & \longrightarrow & M \otimes_R N' \end{array}$$

and $h_M : A \otimes_R C^{op}\text{-mod} \to B \otimes_R C^{op}\text{-mod}$ such that $\operatorname{Hom}_A(tM(N), L) \cong \operatorname{Hom}_B(N, h_M(L))$ is natural in N and L; i.e. t_M is left adjoint to h_M.

Remark 1.1. This is the most general version, but you can safely forget C to get a more readable version of this theorem.

Proof. Let

$$\varphi \mapsto (n \mapsto \underbrace{(m \mapsto \varphi(m \otimes n))}_{\psi_n}).$$

This is a homomorphism of abelian groups. Define $\psi_n : M \to L$ be $\psi_n(m) = m \otimes n$. Then

$$\psi_n(am) = \psi_n((am) \otimes n) = a\psi(m \otimes n) = a\psi_n(m),$$

so $\psi_n \in \operatorname{Hom}_A(M, L)$. Now look at $n \mapsto \psi_n$. Then

$$(b\psi_n)(m) = \psi_n(mb) = mb \otimes n = m \otimes bn = \psi_{bn}(m),$$

so $(n \mapsto \psi_n) \in \operatorname{Hom}_B(N, \operatorname{Hom}_A(M, L))$. Showing that our map is a map of C^{op}-mods is left as an exercise.

Let's find an inverse. Take $\theta \in \mathrm{Hom}_B(N, \mathrm{Hom}(M, L))$, and send

$$\theta \mapsto (m \otimes n \mapsto \theta(n)(m)).$$

Then

$$a(m \otimes n = am \otimes n \mapsto \theta(n)(am) = a\theta(n)(m),$$

so this is a map of A-modules. Also, $(m, n) \mapsto \theta(n)(m)$ gives a map $M \times N \to L$ that is left A-linear, B-balanced, and right C-linear (check this). So $M \otimes_B N \to L$ is a map of $A \otimes_R C^{\mathrm{op}}$-mods. To show that these are inverse maps, let $\varphi \mapsto \theta$, where $\theta(n)(m) = \varphi(m \otimes n)$. Then

$$\theta \mapsto \underbrace{(m \otimes n \mapsto \theta(n)(m) = \varphi(m \otimes n))}_{\varphi}.$$

Check that the other composition works out. $\square$

1.2 Tensor powers and graded algebras

Let M be an R-module, where R is a commutative ring.

Definition 1.1. The k-th **tensor power** of M over R is $M^{\otimes k} = M \otimes_R M \otimes_R \cdots \otimes_R M$.

This satisfies the universal property for multilinear maps:

$$\begin{tikzcd} M \times M \times \cdots \times M \arrow[r] \arrow[d] & L \\ M \otimes_R M \otimes_R \cdots \otimes_R M \arrow[ur, dashed] & \end{tikzcd}$$

Definition 1.2. A **graded ring** $A = \bigoplus_{i=0}^{\infty} A_i$ is ring consisting of a sequence of abelian groups A_i such that

1. The restriction of $+ : A \times A \to A$ to $A_i \times A_i$ is the operation on A_i
2. The restriction of $\cdot : A \times A \to A$ to $A_i \times A_j$ lands in A_{i+j} (so A_0 is a ring).

Here, $\mathrm{gr}^k(A) := A_k$ is called the k-th **graded piece**.

To check that the direct sum of abelian groups together with these maps forms a graded ring, we need these to be the same:

$$(A_i \times A_j) \times A_k \to A_{i+j} \times A_k \to A_{i+k+k},$$

$$A_i \times (A_j \times A_k) \to A_i \times A_{j+k} \to A_{i+j+k}.$$

Definition 1.3. A **graded R-algebra** is a graded ring with the A_i R-algebras, with a map $R \to Z(A_0)$ such that $R \times A_i \to A_i$ and $A_i \times R \to A_i$ are the same, and such that $A_i \times A_j \to A_{i+j}$ is R-bilinear.

Define

$$T(M) = \bigoplus_{k=0}^{\infty} M^{\otimes k},$$

where we have the map $M^{\otimes k} \times M^{\otimes \ell} \to M^{\otimes (k+\ell)}$ given by

$$(m_1 \otimes \cdots \otimes m_k) \cdot (m_1' \otimes \cdots \otimes m_\ell') = m_1 \otimes \cdots m_k \otimes m_1' \otimes \cdots \otimes m_\ell'.$$

Then this is a graded R-algebra.

Example 1.1. Let R be a commutative ring. Then

$$T(R) = \bigoplus_{k=0}^{\infty} R \cong R[x],$$

where the k-th graded piece has basis element $1 \mapsto x^k$.

Example 1.2. Let R be a commutative ring. What is $T(R^{\oplus n}) = T(Rx_1 \oplus \cdots \oplus Rx_n)$? The k-th graded piece is generated by $x_{i_1} \otimes \cdots \otimes x_{i_k}$. However, this is not $R[x_1, \ldots, x_n]$. Notice that $x_i \otimes x_j \neq x_j \otimes x_i$, so $R^{\oplus n} \otimes_R R^{\oplus n} = R^{\oplus n^2}$. So

$$T(R^{\oplus n}) = R\left\langle x_1, \ldots, x_n \right\rangle,$$

the noncommutative polynomial ring in n variables over R.

What is the universal property of T? If $\varphi : M \to A$ is a map of A modules, where A is an R-algebra, then there exists a unique $T(\varphi) : T(M) \to A$ such that

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d] & L \\ T(M) \arrow[ur, dashed, "T(\varphi)"'] & \end{tikzcd}$$

because $T(\varphi)(m_1 \otimes \cdots \otimes m_k) = \varphi(m_1) \otimes \cdots \otimes \varphi(m_j)$ determines $T(\varphi)$.

Let $I = \{m \otimes n - n \otimes m : m, n \in M\}$. Then

$$I = \bigoplus_{k=0}^{\infty} \operatorname{gr}^k(I),$$

where $\operatorname{gr}^k(I) := I \cap \operatorname{gr}^k(T(M))$. Then I is a **graded ideal**. If A is a graded R-algebra and I is a graded ideal of A, then

$$A/I \cong \bigoplus_{k=0}^{\infty} \operatorname{gr}^k(A)/\operatorname{gr}^k(I)$$

is a graded ring.

Definition 1.4. The **symmetric algebra** is $S(M) = T(M)/I$.

In the quotient,

$$m_1 \otimes m_2 \otimes m_3 = m_3 \otimes m_1 \otimes m_2 = m_1 \otimes m_3 \otimes m_2 = \cdots .$$

Example 1.3. $S(R^{\oplus n}) = R[x_1, \ldots, x_n]$.